PH 561.3 Reg. No. [ ][ ][ ][ ][ ][ ]

St Aloysius College (Autonomous)

Mangaluru

Semester III– P.G. Examination – M.Sc. Mathematics

December - 2022

COMPLEX ANALYSIS - I

Time: 3 hrs. **Max Marks: 70**

(14x5=70)

Answer any FIVE FULL questions.

1. a) State and prove Lagrange's inequality in the complex form.

   b) Show that z and z' corresponds to diametrically opposite points on the Riemann sphere if and only if $zz' = -1$

   c) Find the values of $\sqrt[4]{-1}$. **(7+4+3)**

2. a) State and prove the Lucas's theorem.

   b) Prove that $\left|\frac{a-b}{1-\bar{a}\,b}\right| = 1$ if either $|a| = 1$ or $|b| = 1$.What exception must be made if $|a| = |b| = 1$?

   c) Show that $u(x,y) = e^x cosy$ is a harmonic function. **(7+4+3)**

3. a) Let $f(z) = u(z) + iv(z)$ be defined on an open subset U of $\mathbb{C}$ such that u and v have continuous first order partial derivatives. If u and v satisfy C-R equations then prove that $f(z)$ is analytic on U.

   b) Prove that if all the zeros of a polynomial $P(z)$ lie in a half-plane, then all the zeros of the derivative $P'(z)$ also lie in the same half-plane. **(7+7)**

4. a) Define the exponential function e^z and show that e^{iz} has a least positive period 2π and all other periods are integer multiples of 2π.

   b) Show that $e^{a+b} = e^a.e^b \quad \forall\, a, b \in \mathbb{C}$. **(10+4)**

5. a) If $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ is a power series, then show that there exists R with $0 \le R \le \infty$ such that the series converges absolutely for every z with $|z| < R$, the sum of the series is an.analytic function in $|z| < R$.

   b) Prove that a sequence of complex numbers is convergent if and only if it is a Cauchy sequence. **(9+5)**

6. a) Derive a necessary and sufficient condition under which the line integral $\int_\gamma pdx + qdy$ defined in a region Ω depends only on the end points of γ.

   b) Show that there is a unique linear fractional transformation T with $T(z_2) = 1, T(z_3) = 0, T(z_4) = \infty$, where z_2, z_3, z_4 are three distinct points of the extended complex plane. **(10+4)**

7. a) State and prove Cauchy's theorem for a rectangle.

   b) State and prove Morera's theorem. **(9+5)**

8. a) If $\varphi(\xi)$ is continuous on an arc γ, show that $F_n(z) = \int_\gamma \frac{\varphi(\xi)}{(\xi-z)^n}\, d\,\xi$ is analytic in each of the regions determined by γ and its derivative is $F_n'(z) = nF_{n+1}(z)$.

   b) If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is attained on the boundary of E. **(9+5)**

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Semester III – P.G. Examination –M.Sc. Mathematics

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TOPOLOGY

Time : 3 hours

Max. Marks :70

Answer any FIVE FULL questions from the following:

1. a) Define and compare the standard, lower limit and k –topology on $\mathbb{R}$.

   b) If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then prove that the collection $\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$. **(6+8)**

2. a) Define closure of a subset A of a topological space X. Prove that $x \in \bar{A}$ if and only if every neighbourhood of x intersects A.

   b) Let X be a topological space and $A \subseteq X$. Define the interior of A and the boundary of A. Prove that closure of A is the disjoint union of $Int(A)$ and $Bd(A)$. **(7+7)**

3. a) If X is a Hausdorff space, then prove that every finite subset of X is closed in X. Is the converse true? Justify.

   b) Prove that a subspace of a Hausdorff space is Hausdorff.

   c) Prove that product of two Hausdorff spaces is Hausdorff. **(6+3+5)**

4. a) Define open maps and closed maps. Show that a continuous open map need not be closed.

   b) Prove that $\pi : X \times Y \to X$ defined by $\pi((x, y)) = x, \forall\, x \in X$ is continuous.

   c) Let X be a metrizable space. Prove that a map $f: X \to Y$ is continuous if and only if $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X. **(5+2+7)**

5. a) Show that the union of a collection of connected sets in a topological space having a point in common is connected.

   b) Prove that every path connected space is connected.

   c) Show that a topological space is compact if and only if for every collection $\mathcal{C}$ of closed sets in X satisfying the finite intersection property, the intersection $\cap\, \mathcal{C}$ is non-empty. **(4+4+6)**

Contd...

6. Define a homeomorphism. If $f:X \rightarrow Y$ is a bijective continuous map, where X is compact and Y is Hausdorff, then show that f is a homeomorphism.

(14)

7. a) Define a second countable space. If X is second countable, show that every open cover of X contains a countable subcollection covering X.

   b) Define a separable space. Prove that every second countable space is separable.

   c) Prove or disprove: Every second countable space is first countable.

(7+5+2)

8. State and prove the Tietze extension theorem. **(14)**

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St Aloysius College (Autonomous)

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Semester III– P.G. Examination – M.Sc. Mathematics

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NUMERICAL ANALYSIS WITH COMPUTATIONAL LAB

Time: 3 hrs. **Max Marks: 70**

Answer any FIVE FULL questions. **(14x5=70)**

1. a) Apply Newton'-Raphson's method to determine a root of the equation $\cos x - xe^x = 0$. Carry out four iterations.

   b) Derive the secant method to find the root of an equation $f(x)=0$.

   c) Find the real root of the equation $f(x) = x^3 - 5x + 1 = 0$ using the Regula-falsi method correct to three decimal places. **(5+5+4)**

2. a) Derive the Muller method to find the real root of the equation $f(x) = 0$.

   b) Let i) $\propto$ be a root of $f(x) = 0$ which is equivalent to $x = \emptyset(x)$.
   ii) I, be any interval containing the point $x = \propto$.
   iii) $|\emptyset'(x)| < 1$ for all x in I. Then the sequence of approximations $x_0, x_1 \ldots\ldots$ will converge to the root $\propto$ provided the initial approximation x_0 is chosen in I.

   c) Find a real root of the polynomial equation $2x^3 - 5x + 1 = 0$ by Birge-Vieta method. Carry out two iterations. Use initial approximation $p_0 = 0.5$. **(6+4+4)**

3. a) Solve the equations
   $54x + y + z = 110$
   $2x + 15y + 6z = 72$
   $-x + 6y + 27z = 85$
   by Gauss-Seidal iteration method. Carry out four iterations by taking $(0,\ 0,\ 0)^t$ as the initial solution.

   b) Find the largest eigen value and a corresponding eigen vector of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

   Carry out 5 iterations by taking $(1,\ 1,\ 1)^t$ as the initial eigen vector. **(7+7)**

4. a) Derive Gregory-Newton's forward interpolation formula. Also find the truncation error.

   b) Given the following values of $f(x)$ and $f'(x)$

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

   Estimate the values of $f(-0.5)$ and $f(0.5)$ using the hermite interpolation. The exact values are $f(-0.5) = \frac{33}{64}$ and $f(0.5) = \frac{97}{64}$. **(7+7)**

5. a) Derive composite trapezoidal rule. Find its truncation error.

Contd...2

b) Evaluate the integral $I = \int_0^1 \frac{dx}{2x^2+2x+1}$ by using Lobotto 3-point formula.

c) Given the following values find the approximate values of $f'(2.0)$ using linear and quadratic interpolation and $f''(2.0)$ using quadratic interpolation.

i	0	1	2
x_i	2.0	2.2	2.6
f_i	0.69315	0.78846	0.95551

(6+4+4)

6. a) Evaluate $\int_{-2}^{2}\int_0^4 (x^2 - xy + y^2)\,dxdy, \quad h = k = 2$ using the trapezoidal rule.

b) Derive 3-point guass-legendre quadrature formula.

c) Derive the Guass-Hermite one-point formula. **(6+6+2)**

7. a) Solve the IVP $u' = -2tu^2, u(0) = 1, h = 0.2$ on $[0, 1]$ using euler's method.

b) Given the IVP $y' = x + y,\ y(0) = 1,$ on $[0,\ 1]$ with $h = 0.2$. estimate $y(0.4)$ using i) Modified Euler-Cauchy method

ii) Heuns' method. **(6+8)**

8. a) Solve the IVP $y' = 2xy,\ y(0) = 3,$ with $h = 0.2$ in the interval $[0,\ 0.4]$. Use the fourth order classical Runge-Kutta method.

b) Solve the IVP $u' = t + u,\ u(0) = 1$ on $[0, 1]$ with $h = 0.2$ using Adam's Bashforth third order method. **(7+7)**

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St. Aloysius College (Autonomous), Mangaluru

Semester III P. G. Examination - M. Sc. Mathematics

December 2022

Commutative Algebra

Time : 3 Hours

Max. Marks : 70

Answer any FIVE full questions.

1. (a) Define a nilpotent element in a ring. Prove that $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in A[x]$ is nilpotent if and only if a_i is nilpotent for all $0 \leq i \leq n$.

   (b) Define the radical of an ideal in a ring. Prove that the radical of an ideal I is the intersection of all prime ideals which contain I. **(5+9)**

2. (a) If $J(A)$ denotes the Jacobson radical of a ring A then prove that $y \in J(A) \iff 1 - xy$ is a unit in A for every $x \in A$.

   (b) In the ring $A[x]$, show that the Jacobson radical is equal to the nilradical.

   (c) Let $f : A \to B$ be a ring homomorphism. Define extended and contracted ideals. For the ideals I and J in A, Prove that $(I + J)^e = I^e + J^e$ and $r(I)^e \subseteq r(I^e)$. **(5+5+4)**

3. (a) Let $P_1, P_2, \ldots, P_n$ be prime ideals in a ring A. If I is an ideal of A contained $\bigcup_{j=1}^{n} P_j$, then show that $I \subseteq P_j$ for some j.

   (b) Define the prime spectrum Spec(A) of a ring A. Prove that Spec(A) is a compact topological space. **(7+7)**

4. (a) Define annihiltor Ann(M) of an A-module M. For any two A-modules M and N, show that Ann($M + N$) =Ann(M)$\cap$Ann(N).

   (b) State Nakayama's lemma. Let M be a finitely generated A - module, N a submodule of M, I be an ideal of A contained in the Jacobson radical of A. If $M = IM + N$, then show that $M = N$.

   (c) Let M be a finitely generated A - module and $\phi : M \to A^n$ a surjective homomorphism, where $n \in \mathbb{N}$. Show that $\ker\phi$ is finitely generaed. **(5+5+4)**

5. (a) Let $g : A \to B$ be a ring homomorphism, S be a multiplicatively closed subset of A such that $g(s)$ is a unit in B for all $s \in S$. Prove that there exists a ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$.

   (b) Show that the operation S^{-1} is exact.

   (c) Show that the operation S^{-1} commutes with formation of finite sums, intersections and radicals. **(5+4+5)**

contd...2

6. (a) Let $f : A \to B$ be a ring homomorphism and let P be a prime ideal of A. Prove that P is a contraction of a prime ideal of B if $P^{ec} = P$.

   (b) If I is an ideal of a ring A, then show that $S = 1 + I$ is a multiplicatively closed subset of A. Further deduce that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}A$. **(9+5)**

7. (a) Define a primary ideal of a ring A. Is every primary ideal prime? Justify your answer.

   (b) Let Q be a P-primary ideal, x an element of A. Then prove that
   (i) If $x \in Q$ then $(Q : x) = 1$.
   (ii) If $x \notin Q$ then $(Q : x)$ is p-primary and $r(Q : x) = P$.
   (iii) If $x \notin P$ then $(Q : x) = Q$.

   (c) State and prove the first uniqueness theorem for primary decomposition. **(2+6+6)**

8. (a) Give an example for an A-module which satisfies d.c.c. but not a.c.c.

   (b) Prove that a ring A is Noetherian if and only if the polynomial ring $A[x]$ is Noetherian. **(2+12)**

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