PH 561. 4

St. Aloysius College (Autonomous), Mangaluru

Reg. No. | | | | | | |

Semester IV - P. G. Examination - M. Sc. Mathematics

July 2022

Measure Theory and Integrtion

Time : 3 Hours

Max. Marks : 70

Answer any FIVE full questions.

1. (a) If $A \subseteq \mathbb{R}$, and $k \in \mathbb{R}$, show that $m^*(kA) = |k|\, m^*(A)$, where $kA = \{kx \ : \ x \in A\}$.

   (b) Show that Lebesgue outer measure is countably subadditive.

   (c) Show that for any $A \subseteq \mathbb{R}$ with $m^*(A) < +\infty$, and $\epsilon > 0$, there is an open set O containing A such that $m^*(O) < m^*(A) + \epsilon$.

   (5+5+4)

2. (a) Prove or disprove: If $A \subseteq \mathbb{R}$ with $m^*(A) = 0$, then A is countable.

   (b) Show that every interval is Lebesgue measurable, and hence prove that every Borel set is Lebesgue measurable.

   (5+9)

3. (a) Define a Lebesgue measurable function. If f is a Lebesgue measurable function and $f = g$ a. e. on E, then show that g is Lebesgue measurable.

   (b) Let E be a Lebesgue mearurable subset of $\mathbb{R}$, and $c \in \mathbb{R}$. If $f, g : E \to \mathbb{R}$ are Lebesgue measurable functions, show that cf, $f + g$ and fg are Lebesgue measurable.

   (c) Prove that a real valued monotonically decreasing function on a Lebesgue measurable set is Lebesgue measurble.

   (4+6+4)

4. (a) Define the Lebesgue integral of a non-negative Lebesgue measurable function f. Prove that $\int f\, dx = 0$ if and only if $f = 0$ a. e. on E.

   (b) If a bounded function $f : [a,\ b] \to \mathbb{R}$ is Riemann integrable, then prove that f is Lebesgue integrable. Is the converse true? Justify.

   (6+8)

5. (a) State and prove Fatou's Lemma.

   (b) Let $\{f_n\}$ be a monotonically increasing sequence of non-negative Lebesgue measurable functions converging to a function f. Prove that $\int f\, dx = \lim \int f_n\, dx$.

   (10+4)

contd...2

6. (a) Define a convex function. Prove that a convex function on an open interval is continuous.
   (b) State and prove Jensen's inequality. (7+7)

7. Show that L^p spaces, $1 \leq p \leq \infty$, are complete. (14)

8. (a) Define the notions of
   (i) a measurable space,
   (ii) a signed measure on a measurable space.
   (b) Show that if $\phi(E) = \int_E f\, d\mu$, where $\int f\, d\mu$ is defined, then ϕ is a signed measure.
   (c) Define a positive set with respect to a signed measure ν. Prove that a countable union of sets positive with respect to a signed measure ν is a positive set. (3+4+7)

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St. Aloysius College (Autonomous), Mangaluru

Semester IV – P.G. Examination – M.Sc. Mathematics

July - 2022

COMPLEX ANALYSIS - II

Time : 3 Hours | Max. Marks : 70

Answer any **FIVE FULL** questions from the following: (14 × 5 = 70)

1. (a) Define a simply connected region. Give two examples.

   (b) Prove that a region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a which do not belong to Ω. (2+12)

2. (a) State and prove the Residue theorem.

   (b) Find a homology basis for the annulus defined by $r_1 < |z| < r_2$.

   (c) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z| = 2$. (8+4+2)

3. (a) State and prove the Argument Principle.

   (b) State and prove the Rouche's theorem.

   (c) Evaluate $\int_C \frac{3z^2 + z - 1}{(z^2 - 1)(z-3)} dz$, where $C : |z| = 2$.

   (d) Evaluate $\int_C \frac{1}{(z-1)^2(z-3)} dz$, where $C : |z| = 2$. (5+5+2+2)

4. (a) If u_1 and u_2 are harmonic in a region Ω, then prove that $\int_\gamma u_1^* du_2 - u_2^* du_1 = 0$ for every cycle γ which is homologous to zero in Ω.

   (b) State and prove the mean value property for harmonic functions. (7+7)

5. (a) Prove that a non-constant harmonic function defined in a region Ω has neither a maximum nor a minimum in Ω.

   (b) Let Ω^+ denotes the part in the upper half plane of a symmetric region Ω, and let σ be the part of real axis in Ω. Suppose that $v(x)$ is real and continuous in $\Omega^+ \cup \sigma$, harmonic in Ω^+ and zero on σ. Prove that v has a harmonic extension to Ω which satisfies the symmetric relation $v(\bar{z}) = -v(z)$. (7+7)

contd...2

6. (a) Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω uniformly on every compact subset of Ω. Prove that $f(z)$ is analytic in Ω.

(b) State and prove the Hurwitz theorem.

(7+7)

7. (a) Show that $\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z$ uniformly on every compact subset of the complex plane.

(b) If $f(z)$ is analytic in a region Ω containing a, then show that the representation $f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \cdots$ is valid in the largest open disc centered at a and contained in Ω.

(7+7)

8. (a) Show that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

(b) Prove that $f(z)$ is an entire function without zeroes if and only if $f(z) = e^{g(z)}$, where $g(z)$ is an entire function.

(c) Prove that a necessary and sufficient condition for the absolute convergence of the product $\prod_{n=1}^{\infty}(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty}|a_n|$.

(6+3+5)

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PS 564.4

Reg. No. | | | | | | |

St Aloysius College (Autonomous)
Mangaluru
Semester IV - P.G. Examination - M.Sc. Mathematics
July - 2022
FUNCTIONAL ANALYSIS

Time: 3 hrs.

Max Marks: 70

Answer any FIVE FULL questions from the following :

1. a) State and prove the Cantor's intersection theorem. Show that the theorem fails if the closed sets in the hypothesis are replaced by open sets.

   b) Prove that every complete metric space is of second category. (7+7)

2. a) State and prove the Holder's inequality for $n-tuples$ of scalars and deduce the Minkowski's inequality.

   b) Let p be a real number such that $p \geq 1$. Prove that the linear space $l_p{}^n$ of all $n-tuples$ $x = (x_1, x_2, \ldots, x_n)$ of scalars forms a Banach space with respect to the norm given by $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$. (8+6)

3. a) Let N be a finite dimensional normed linear space with dimension $n > 0$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis of N. Show that the map $T: N \to l_1{}^n$ given by $T(x) = (x_1, x_2, \ldots, x_n)$, whenever $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$, is continuous.

   b) Let L be a linear space made into a normed linear space by $\|.\|$ and $\|.\|'$. Show that these two norms are equivalent if and only if there exist positive reals K_1 and K_2 such that $K_1\|x\| \leq \|x\|' \leq K_2\|x\|$, for all $x \in L$. (9+5)

4. a) Let N be a nonzero normed linear space. Prove that N is a Banach space if and only if $\{x \in N: \|x\| = 1\}$ is complete as a subspace of N.

   b) Define the conjugate space N^* of a normed linear space N. Prove that there is an isometric isomorphism of N into N^{**}. (7+7)

5. State and prove the open mapping theorem. (14)

6. a) State and prove the parallelogram law in a Hilbert space H.

   b) Define Hilbert space. Prove that a complex Banach space B is a Hilbert space if and only if the parallelogram law holds in B. (3+11)

7. a) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a nonzero vector z_0 in H such that $z_0 \perp M$.

   b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M + N$ is also closed.

   c) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$. (5+6+3)

Contd...2

8. a) For an orthonormal set $\{e_1, e_2, \dots, e_n\}$ in a Hilbert space H and $x \in H$ show that

i) $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$

ii) $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j$, for each $j, 1 \leq j \leq n$.

b) Let H be a Hilbert space, prove that for each $T \in B(H)$ there exists a unique operator $T^* \in B(H)$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in H$.

c) If T is an operator on a Hilbert space H such that $\langle Tx, x \rangle = 0$ for all $x \in H$, then show that $T = 0$.

(5+6+3)

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PS 565.4

Reg. No. [| | | | |]

St Aloysius College (Autonomous)
Mangaluru
Semester IV - P.G. Examination - M.Sc. Mathematics
July - 2022
PARTIAL DIFFERENTIAL EQUATIONS

Time: 3 hrs.

Max Marks: 70

1. a) Show that the general solution of the Lagrange's equation in two independent variables x, y for a single unknown function $u(x,y)$, $P(x,y,u)\frac{\partial u}{\partial x}+Q(x,y,u)\frac{\partial u}{\partial y}=R(x,y,u)$ is $F(\varphi(x,y,u),\psi(x,y,u))=0$ where F is an arbitrary function and $\varphi(x,y,u)=c_1$, $\psi(x,y,u)=c_2$ are two independent first integral curves of the equation $\frac{dx}{P(x,y,u)}=\frac{dy}{Q(x,y,u)}=\frac{du}{R(x,y,u)}$.

   b) Find the general integral of the quasi-linear partial differential equation $(2xy-1)p+(z-2x^2)q=2(x-yz)$.

(7+7)

2. a) Solve $(y^2+yz+z^2)dx+(z^2+zx+x^2)dy+(x^2+xy+y^2)dz=0$

   b) Test for integrability of $z(y^2+z)dx+z(z+x^2)dy-xy(x+y)dz=0$ and find its primitive.

(7+7)

3. a) Find the orthogonal trajectories on the cone $x^2+y^2=z^2\tan^2\alpha$ of the curves in which it is cut by the system of planes $z=c$.

   b) Obtain the partial differential equation by eliminating the arbitrary function f from the equation $u=(x-y)f(x^3+y^3)$.

   c) Find the complete integral of the partial differential equation $p^2y(1+x^2)=qx^2$.

(7+3+4)

4. a) When are two first order partial differential equations $f(x,y,u,p,q)=0$ and $g(x,y,u,p,q)=0$ said to be compatible. Derive a necessary condition for their compatibility.

   b) Find the characteristics of the equation $2pq-u=0$ and find the integral surface satisfying $u(0,y)=\frac{y^2}{2}$.

(7+7)

5. a) Find the complete integral of the partial differential equation $(p^2+q^2)x=pz$ and deduce the solution which passes through the curves $x=0$ and $z^2=4y$.

   b) Find the surface which intersects surfaces of the system $z(x+y)=c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2=1,\ z=1$.

(7+7)

6. a) Solve the equation $(D^2+2DD'+D'^2-2D-2D')u=\sin(x+2y)$

   b) Solve the equation $(D^2+3DD'+2D'^2)u=x+y$

(7+7)

7. a) Classify and reduce the equation $u_{xx}+x^2u_{yy}=0$ to a canonical form and solve it.

   b) Find the characteristic curves of the partial differential equation $3u_{xx}+10u_{xy}+3u_{yy}=0$

(8+6)

Contd...2

8. a) Obtain the solution of the wave equation $u_{tt} - 4u_{xx} = 0$, $0 < x < 1$, $t > 0$ subject to the boundary conditions $u(x,0) = x - x^2$, $\frac{\partial u}{\partial t}(x,0) = \sin \pi x \quad 0 \leq x \leq 1$ and $u(0,t) = 0$, $u(1,t) = 0$, $t \geq 0$.

b) Obtain the solution of the one dimensional diffusion equation in the region $0 \leq x \leq \pi$, $t \geq 0$ subject to

(i) T remains finite as $t \to \infty$,

(ii) $T = 0$ if $x = 0$ and $x = \pi$, $\forall t$ and

(iii) At $t = 0$ $T = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$

(7+7)

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PS 566.4

Reg. No.

St Aloysius College (Autonomous) Mangalore

Semester IV - P. G. Examination - M. Sc. Mathematics

July 2022

Algebraic Number Theory

Time : 3 Hours | Max. Marks : 70

Note: Answer any FIVE full questions.

1. (a) Show that an integer n is divisible by 9 if and only if sum of its digits in its decimal expansion is divisible by 9.

   (b) Solve the congruence $6x \equiv 12 \pmod 9$.

   (c) Define Euler totient function $\varphi(n)$. Let a, m, n are positive integers with $(a, m) = 1$. Prove that $a^{\varphi(m)} = 1 \pmod m$. Further if p is prime number, then prove that $1^p+2^p+\cdot\cdot+(p-1)^p \equiv 0 \pmod p$.

   (3+4+7)

2. (a) Let p be an odd prime and a be any integer, show that $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod p$.

   (b) For an odd prime p show that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

   (c) State and prove Wilson's Theorem. (6+4+4)

3. (a) State and prove Gauss lemma.

   (b) Find all prime p such that $x^2 \equiv 13 \ (mod\ p)$ has a solution. (10+4)

4. (a) If $\alpha \in \mathbb{R}$ is an algebraic number of degree $n > 1$, then prove that there exists a constant $c(\alpha) > 0$, such that for any rational number $\frac{p}{q}$ with $\gcd(p, q) = 1$, $q > 0$, thte following inequality holds.

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha)}{q^n}.$$

   (b) If $\alpha \in \mathbb{C}$ and there is a finitely generated non-zero $\mathbb{Z}$-submodule M of $\mathbb{C}$ such that $\alpha M \subseteq M$, then prove that α is an algebraic integer.

   (c) If K is an algebraic number field of degree n, then prove that there exists exactly n distinct $\mathbb{Q}$-embeddings of K into $\mathbb{C}$. (5+4+5)

contd...2

5. (a) Let K be an algebraic number field. If $\alpha \in K$ is an algebraic integer, then prove that $Tr_{K/\mathbb{Q}}(\alpha)$ and $Nr_{K/\mathbb{Q}}(\alpha)$ are integers.

(b) Let K be an algebraic number field and $\mathcal{O}_K$ be the ring of integers of K. Then prove that

i) $Nr_{K/\mathbb{Q}}(\alpha) = \pm 1$ if and only if α is a unit in $\mathcal{O}_K$.

ii) If $\alpha, \beta \in \mathcal{O}_K$ are associates, then $Nr_{K/\mathbb{Q}}(\alpha) = \pm Nr_{K/\mathbb{Q}}(\beta)$.

(c) Prove that every algebraic number field has an integral basis (3+4+7)

6. (a) If d is a square free integer, then find an integral basis and discriminant of $K = \mathbb{Q}(\sqrt{d})$.

(b) If $\zeta = e^{\frac{2\pi i}{5}}$ and $K = \mathbb{Q}(\zeta)$, then find $Tr_{K/\mathbb{Q}}(\zeta + \zeta^2 + \zeta^3)$. (11+3)

7. (a) Let d be square free integer less than -11 and $d \equiv 1 \pmod 4$. Then prove that, the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{d})$ is not a Euclidean domain.

(b) Prove or disprove the following:

i) Every Unique factorization domain is integrally closed.

ii) Every principal ideal domain is a Dedekind domain.

iii) Every Dedekind domain is a principal ideal domain.

(4+10)

8. Let $\mathcal{O}_K$ be ring of integers of an algebraic number field K. Prove that every non-zero proper ideal of $\mathcal{O}_K$ can be uniquely written as product of finitely many non-zero prime ideals of $\mathcal{O}_K$. (14)

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