PH 561.4

Reg. No. :

St Aloysius College (Autonomous)

Mangaluru

Semester IV – P.G. Examination – M.Sc. MATHEMATICS

APRIL - 2019

MEASURE THEORY AND INTEGRATION

Time: 3 Hours Max. Marks: 70

Answer any FIVE FULL questions from the following: (5x14=70)

1.a) Define the Lebesgue outer measure of a subset of $\mathbb{R}$. Prove that Lebesgue outer measure is countably subadditive.

b) Show that, for any set A and any $\epsilon > 0$, there is an open set O containing A such that $m^*(O) \leq m^*(A) + \epsilon$.

c) Define a Lebesgue measurable set. Show that if $m^*(E) = 0$, then every subset of E is Lebesgue measurable. Further, if F is Lebesgue measurable and $m^*(F \Delta G) = 0$, then show that G is Lebesgue measurable. (5+4+5)

2.a) Show that every interval is Lebesgue measurable and hence prove that every Borel set is measurable.

b) For $k > 0$ and $A \subseteq \mathbb{R}$ let $kA = \{kx \mid x \in A\}$. Show that $m^*(kA) = km^*(A)$. (11+3)

3.a) Prove that the following statements regarding the set $E \subseteq \mathbb{R}$ are equivalent:

i) E is Lebesgue measurable.

ii) For all $\epsilon > 0$, there exists O, an open set, $O \supseteq E$ such that $m^*(O - E) \leq \epsilon$.

iii) There exists G, a $G_\delta - set$, $G \supseteq E$ such that $m^*(G - E) = 0$.

b) Define a Lebesgue measurable function. If $\{f_n\}$ is a sequence of Lebesgue measurable functions defined on the same measurable set, then show that '$\inf f_n$' is Lebesgue measurable. (10+4)

4.a) Define a simple function, and the Lebesgue integral of a simple function. If $a_1, a_2, \dots a_n$ are the distinct values taken by a measurable simple function ϕ and if $A_i = \{x / \phi(x) = a_i\}, 1 \leq i \leq n$ then prove that

i) $\int_E \phi \, dx = \sum_{i=1}^{n} a_i m(A_i \cap E)$ for any measurable set E.

ii) $\int a\phi \, dx = a \int \phi \, dx$ for all $a > 0$.

b) Define the Lebesgue integral of a non-negative measurable function. Show that if f is a non-negative measurable function, then $f = 0$ $a.e.$ if and only if, $\int f \, dx = 0$. (7+7)

Contd...2

5.a) State Fatou's lemma. Let $\{f_n, n = 1,2,3,\ldots\}$ be a sequence of non-negative measurable functions such that $\{f_n(x)\}$ is monotone increasing for each x. If $f = \lim f_n$, then prove that $\int f dx = \lim_{n\to\infty} \int f_n dx$.

b) Let f be a non-negative measurable function. Then prove that there exists a sequence $\{\phi_n\}$ of measurable simple functions such that $\int f dx = \lim_{n\to\infty} \int \phi_n dx$.

c) If f and g are non-negative measurable functions, prove that $\int (f+g)dx = \int f\,dx + \int g\,dx$.

(3+6+5)

6.a) Define a measure space $[|X, S, \mu|]$ and $L^1(X,\mu)$.
Show that $L^\infty(X,\mu)$ is a vector space over the real numbers.

b) Define a convex function Ψ on an interval (a,b). Prove that a convex function on (a,b) is continuous.

(5+9)

7.a) State and derive Holder's inequality.

b) Let f and g be non-negative measurable functions. Show that equality occurs in Holder's inequality if, and only if, $sf^p + tg^q = 0$ a.e. for some constants s and t not both zero.

(7+7)

8.a) Define a signed measure on a measure space $[|X, S|]$.
When do you say that a set $A \subseteq X$ is a positive set with respect to a signed measure γ on $[|X, S|]$. Prove that a countable union of positive sets with respect to a signed measure γ is a positive set.

b) Show that if $\phi(E) = \int_E f\,d\mu$, where $\int f\,d\mu$ is defined, then ϕ is a signed measure.

(9+5)

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Semester IV – P.G. Examination – M.Sc. Mathematics

April - 2019

COMPLEX ANALYSIS II

Time: 3 hrs.

Max Marks: 70

Answer any **FIVE** FULL questions from the following: (14×5=70)

1. a) Define a simply connected region. Give two examples.

   b) Prove that a region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points $'a'$ which do not belong to Ω.

(2+12)

2. a) State and prove the residue theorem.

   b) Find a homology basis for the annulus defined by $r_1 < |z| < r_2$.

   c) Evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = 2$.

(8+4+2)

3. a) Find the value of $\int_{|z|=1} \cot z \, dz$.

   b) State and prove the argument principle.

   c) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2 ?

   d) Evaluate $\int_0^{\infty} \frac{\sin x}{x(1+x^2)} dx$.

(2+5+2+5)

4. a) If u_1 and u_2 are harmonic in a region Ω, then prove that $\int_\gamma u_1 \, {}^*du_2 - u_2 \, {}^*du_1 = 0$ for every cycle γ which is homologous to zero in Ω.

   b) State and prove the mean-value property for harmonic functions.

(7+7)

5. a) Prove that a non-costant harmonic function defined in a region Ω has neither a maximum nor a minimum in Ω.

   b) If $f(z)$ is analytic in $|z| < 1$ and satisfies $|f| = 1$ on $|z| = 1$, then show that $f(z)$ is rational.

   c) State Poisson's formula.

(7+5+2)

Contd...2

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6. a) Suppose that $f_n(z)$ is analytic in the region Ω_n and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω, uniformly on every compat subset of Ω, then prove that $f(z)$ is analytic in Ω.

b) Show that $\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z$ uniformly on every compact subset of the complex plane.

(7+7)

7. a) If $f(z)$ is analytic in a region containing $'a'$, then show that the representation $f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$ is valid in the largest open disc centered at $'a'$ and contained in Ω.

b) Expand $f(z) = \frac{z}{(z-1)(z-2)}$ in a Laurent's series, valid for $1 < |z| < 2$.

(10+4)

8. a) Prove that a necessary and sufficient condition for the absolute convergence of the product $\prod_{n=1}^{\infty}(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty}|a_n|$.

b) Prove that every function which is meromorphic in the whole plane is the quotient of two entire functions.

c) Evaluate $\Gamma\left(\frac{1}{2}\right)$.

(7+5+2)

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Mangaluru
Semester IV – P.G. Examination - M. Sc. Mathematics
April - 2019

FUNCTIONAL ANALYSIS

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) State and prove the Cantor's intersection theorem in metric spaces.
   b) State and prove Baire's category theorem. **(7+7)**

2. a) Prove that the following are equivalent for a linear transformation T from a normed linear space N into a normed linear space N'.
   i) T is continuous

   ii) T is continuous at the origin.
   iii) There exists a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\|$ for every $x \in N$.
   iv) If $S = \{x \in N : \|x\| \leq 1\}$ then $T(S)$ is a bounded set in N'.

   b) Prove that a nonzero normed linear space N is a Banach space if and only if $\{x \in N : \|x\| = 1\}$ is complete. **(7+7)**

3. a) Prove that the set of all bounded linear transformations of a normed linear space into a Banach space is itself a Banach space.
   b) Let $\mathcal{B}(N)$ denote the algebra of operators on a normed linear space N. if $T_n \to T$ and $T_n' \to T'$ in $\mathcal{B}(N)$, then show that $T_n T_n' \to TT'$ in $\mathcal{B}(N)$.
   c) Let N and N' be normed linear spaces. Define an isometric isomorphism of N into N'. If $T : N \to N'$ is an isometric isomorphism of N onto N', then show that $T^{-1} : N' \to N$ is also an isometric isomorphism. **(10+2+2)**

4. State and prove the open mapping theorem. **(14)**

5. a) Let B be a Banach space and M, N be closed linear subspaces of B such that $B = M \oplus N$. Then prove that there exists a projection P on B such that M is the range of P and N is the null space of P.
   b) Let B be a Banach space and N be a normed linear space. If $\{T_i\}_{i \in I}$ is a nonempty set of continuous linear transformations of B into N such that $\{T_i(x)\}_{i \in I}$ is a bounded subset of N for each $x \in B$, then prove that $\{T_i\}_{i \in I}$ is a bounded subset of $\mathcal{B}(B, N)$. **(7+7)**

Contd...2

6. a) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a nonzero vector z_0 in H such that $z_0 \perp M$.

   b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then prove that $M + N$ is also closed.

   c) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$.

(5+5+4)

7. a) Let $\{e_i\}_{i\in I}$ be an orthonormal set in a Hilbert space H. Show that for any vector x in H, the set $\{e_i : \langle x, e_i\rangle \neq 0\}$ is atmost countable and prove that $\sum_{i\in I} |\langle x, e_i\rangle|^2 \leq \|x\|^2$.

   b) Let H be a Hilbert space. Prove that for each operator $T \in \mathcal{B}(H)$ there exists a unique operator $T^* \in \mathcal{B}(H)$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, for all $x, y \in H$.

(7+7)

8. State and prove the spectral theorem for a finite dimensional Hilbert space.

(14)

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Semester IV – P.G. Examination – M.Sc. MATHEMATICS

APRIL - 2019

PARTIAL DIFFERENTIAL EQUATIONS

Time: 3 Hours

Max. Marks: 70

Answer any FIVE FULL questions from the following: (14x5=70)

1.a) Prove that a necessary and sufficient condition for a differential equation $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$ to be integrable is that $X.\,curl\,X = 0$.

b) Find the integral curves of the equations

$$\frac{dx}{e^{-y}\cos x} = \frac{dy}{e^{-y}\sin x} = dz$$

c) Test for integrability of

$(y+z)dx + (z+x)dy + (x+y)dz = 0$ and find its primitive. (7+3+4)

2.a) Define orthogonal trajectories. Find the orthogonal trajectories on the cone $(x+y)z = 1$ of the conics in which it is cut by the system of planes $x - y + z = K$, where K is a parameter.

b) Solve $(y^2 + yz)\,dx + (xz + z^2)\,dy + (y^2 - xy)\,dz = 0$. (7+7)

3.a) Obtain the partial differential equation for the function $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$, where f is an arbitrary function.

b) Find the integral surface of the linear partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ containing the straight line $x + y = 0, z = 1$.

c) Find the complete integral of $pqu = p^2(xq + p^2) + q^2(yp + q^2)$ (4+7+3)

4.a) Find the characteristic of the equation $pq = z$ and determine the integral surface which passes through the straight line $x = 1, z = y$.

b) Derive a necessary condition for the compatibility of $f(x,y,u,p,q) = 0$ and $g(x,y,u,p,q) = 0$. (8+6)

5.a) Find the surface which is orthogonal to the one parameter family of system $x^2 + y^2 + u^2 = Ku$.

b) Find the general solution of $(2x - y)y^2\,u_x + 8x^2(y - 2x)\,u_y = 2(4x^2 + y^2)\,u$ and deduce the solution of Cauchy problem $u(x,0) = \frac{1}{2x}$ on the portion of the $x - axis$. (6+8)

6.a) Solve $(D^2 + DD' - 6D'^2)u = y\cos x$.

b) Solve $(D^2 - D')u = e^{2x+y}$.

c) Solve $(D^2 - DD' + D' - 1)u = \cos(2x + y)$. (7+2+5)

7.a) Give the classification of a second order semilinear partial differential equation in two independent variables for a single unknown function $u(x, y)$. Give the canonical forms of the transformed equations. In the hyperbolic case describe the method of reducing it to the canonical form.

b) Find the characteristics of the equation $u_{xx} + 2u_{xy} + \sin^2 x\, u_{yy} + u_y = 0$ when it is of hyperbolic type. (12+2)

8.a) Classify the following equation and reduce it to canonical form $y^2 u_{xx} - x^2 u_{yy} = 0$.

b) Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi,\ t \ge 0$ Subject to the conditions –

i) T remains finite as $t \to \infty$

ii) $T = 0$ if $x = 0$ and $x = \pi$, $\forall t$

iii) At $t = 0$, $T = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$ (7+7)

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St Aloysius College (Autonomous)

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Semester IV – P.G. Examination - M. Sc. Mathematics

April - 2019

ALGEBRAIC NUMBER THEORY

Time: 3 Hours | Max. Marks: 70

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) Prove that Euler's phi-function $\phi(n)$ is a multiplicative function.

   b) Prove that $a^{\phi(m)} \equiv 1 \, (mod\, m)$, if $\gcd(a,m)=1$.

   c) Solve the congruence $5x \equiv 3 (mod\, 24)$. **(7+4+3)**

2. a) State and prove Lagrange's theorem. Use it to prove Wilson's theorem.

   b) Find all primes p such that the Legendre symbol $(-3|p)=1$. **(10+4)**

3. a) If p is an odd prime and $n \in \mathbb{Z}$, show that the Legendre symbol

   $(n|p) \equiv n^{\frac{p-1}{2}} (mod\, p)$.

   b) State and prove Gauss Lemma.

   c) State quadratic reciprocity law. Let q be an odd prime. Prove that, if $q \equiv 1 (mod\, 4)$ then q is quadratic residue $\bmod p$ if and only if $p \equiv r (mod\, q)$, where r is quadratic residue $mod\, q$. **(4+6+4)**

4. a) Define an algebraic number and an algebraic integer. Show that a rational number is an algebraic integer if and only if it is an integer.

   b) Let K be an algebraic number field and $[K:\mathbb{Q}]=n$. If $\alpha \in K$ and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the distinct $\mathbb{Q}$-isomorphisms from K into $\mathbb{C}$, then prove that

   i) $Tr_{K/Q}(\alpha) = \sigma_1(\alpha) + \sigma_2(\alpha) + \ldots.. + \sigma_n(\alpha)$

   ii) $Norm_{K/Q}(\alpha) = \sigma_1(\alpha)\sigma_2(\alpha)\ldots.\sigma_n(\alpha)$.

   Hence deduce that Trace is additive function and Norm is multiplicative function. **(6+8)**

5. a) For any algebraic number field K, prove that the discriminant $d_{K/Q}$ is congruent to 0 or $1 (\text{mod}\, 4)$.

   b) Determine the discriminant of $\mathbb{Q}(\sqrt{d})$, where d is a square free integer. **(10+4)**

Contd...2

6. a) Let $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. Show that

$$O_K = \begin{cases} \mathbb{Z} + \mathbb{Z}(\sqrt{d}), & \text{if } d \equiv 2 \text{ or } 3 \pmod 4 \\ \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{d}}{2}\right), & \text{if } d \equiv 1 \pmod 4 \end{cases}$$

b) Let $K = \mathbb{Q}(\sqrt{d})$, where $d < 0$, square free integer and $d \equiv 3(mod\,4)$.

Show that O_K is UFD if and only if $d = -1$. **(10+4)**

7. a) Prove that every PID is a Dedekind domain. Is the converse true? Justify.

b) Show that every non-zero ideal of O_K contains a product of finitely many non-zero prime ideals of O_K. **(7+7)**

8. a) Define a fractional ideal of O_K. Show that sum and product of two fractional ideals are again fractional ideals.

b) Let $\wp$ be a prime ideal of O_K and $\wp^{-1} = \{\alpha \in K \mid \alpha\wp \subseteq O_K\}$. Show that $\wp^{-1}$ is a fractional ideal of O_K, $O_K \subsetneq \wp^{-1}$ and $\wp\wp^{-1} = O_K$. **(4+10)**

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Semester IV – P.G. Examination - M. Sc. Mathematics

April - 2018

MEASURE THEORY AND INTEGRATION

Time: 3 Hours Max. Marks: 70

Answer any FIVE FULL questions from the following (14x5=70)

1. a) Let E be a Lebesgue measurable subset of $\mathbb{R}$. Prove the following:

   i) $m^*(E) = m^*(E+y)$

   ii) $E+y = \{x+y : x \in E\}$ is measurable.

   b) Show that the class of all Lebesgue measurable subsets of $\mathbb{R}$ is a σ - algebra. (5+9)

2. a) Given a subset A of $\mathbb{R}$ and $\varepsilon > 0$, prove that

   i) there exists an open set O such that $A \subseteq O$ and $m^*(O) \leq m^*(A) + \varepsilon$.

   ii) there exists a G_δ set G such that $A \subseteq G$ and $m^*(A) = m^*(G)$.

   b) Show that every interval is measurable.

   c) Show that $[0,1]$ is uncountable. ((3+3)+6+2)

3. Let f be a measurable function, prove the following:

   i) If $f = g$ *a.e.*, then g is measurable.

   ii) $f \leq ess \sup f$ *a.e.*

   iii) If B is a borel set, then $f^{-1}(B)$ is a measurable set. (3+5+6)

4. a) State and prove Lebesgue's monotone convergence theorem.

   b) If f and g are non-negative measurable functions and if c is a non-negative real number, then show that $\int (cf+g)dx = c\int f\,dx + \int g dx$

   (4+10)

5. a) If f is Riemann integrable and bounded over the finite interval $[a,b]$ then prove that f is integrable and $R\int_a^b f dx = \int_a^b f dx$. Is the converse holds? Justify.

   b) Show that $\int_1^\infty \frac{dx}{x} = \infty$. (12+2)

6. a) Prove that convex function defined on an open interval is continuous.

   b) State and prove Holder's inequality and discuss the case of equality.

   (6+8)

7. Show that L^p spaces $(1 \leq p \leq \infty)$ are complete. (14)

8. a) Show that if $\phi(E) = \int_E f\,d\mu$ where $\int f\,d\mu$ is defined, then ϕ is a signed measure.

   b) State and prove Jordan decomposition theorem (2+12)

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Semester IV – P.G. Examination – M.Sc. Mathematics

April - 2018

COMPLEX ANALYSIS II

Time: 3 hrs.

Max Marks: 70

Answer any FIVE FULL questions from the following: **(14×5=70)**

1. a) Prove that a region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points $'a'$ which do not belong to Ω.

   b) Is $\Omega = \mathbb{C} - \{a\}$ simply connected? Justify your answer.

(12+2)

2. a) Define the residue of $f(z)$ at an isolated singularity $'a'$.

   b) State and prove the Residue theorem.

   c) State and prove the Argument principle.

(2+7+5)

3. a) Evaluate $\int_c \frac{3z^2+z-1}{(z^2-1)(z-3)} dz$ where C is the circle $|z| = 2$.

   b) Let $(z) = \frac{z^2+1}{(z^2+2z+2)^2}$. Evaluate $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} dz$, where C is the circle $|z| = 4$.

   c) State and prove Rouche's theorem.

   d) Evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5} dx$.

(2+2+5+5)

4. a) State and prove the mean-value property for harmonic functions.

   b) Let u be a bounded harmonic function in $0 < |z| < \rho$. Show that the origin is a removable singularity in the sense that u can be extended to a harmonic function in $|z| < \rho$ when $u(0)$ is properly defined.

(7+7)

5. a) Prove that a nonconstant harmonic function defined in a region Ω has neither a maximum nor a minimum in Ω.

   b) If Ω^+ denotes the part of the upper-half plane of a symmetric region Ω, σ is the part of real-axis in Ω and $v(z)$ is real and continuous in $\Omega^+ \cup \sigma$, harmonic in Ω^+ and zero on σ, then show that v has a harmonic extension to Ω which satisfies the symmetry relation $v(\bar{z}) = -v(z)$.

(7+7)

Contd...2

6. a) If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$ uniformly on every compact subset of Ω, then prove that $f(z)$ is either identically zero or never equal to zero in Ω.

b) Show that $\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z$ uniformly on every compact subset of the complex plane.

(7+7)

7. a) If $f(z)$ is analytic in a region containing $'a'$, then show that the representation $f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$ is valid in the largest open disc centered at $'a'$ and contained in Ω.

b) Find the Laurent series expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ valid in the annular region $2 < |z| < 3$.

(10+4)

8. a) Prove that the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ with $1 + a_n \neq 0$ converges simultaneously with the series $\sum_{n=1}^{\infty}\log(1+a_n)$ whose terms represent the values of the principal branch of logarithm.

b) Evaluate $\Gamma\left(\frac{1}{2}\right)$.

(12+2)

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St Aloysius College (Autonomous)
Mangaluru
Semester IV – P.G. Examination - M. Sc. Mathematics
April - 2018

FUNCTIONAL ANALYSIS

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) State and prove the Cantor's intersection theorem in metric spaces.

   b) Define a nowhere dense set in a metric space. Let $\{A_n\}$ be a sequence of nowhere dense sets in a complete metric space X. Prove that there exists a point in X which is not in any of the A_n's **(7+7)**

2. a) Define a Banach space. Let p be a real number such that $1 \le p < \infty$. Prove that the linear space l_p of all sequences $x = (x_1, x_2, \ldots)$ of scalars such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$, with respect to the norm defined by $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$ is a Banach space.

   b) Prove that a nonzero normed linear space N is a Banach space if and only if $\{x \in N : \|x\| = 1\}$ is complete as a subspace of N. **(7+7)**

3. a) Let N be a finite dimensional normed linear space with dimension $n > 0$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis for N. Let a map $T : N \to l_1^n$ be defined by $T(x) = (x_1, x_2, \ldots, x_n)$ whenever $x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$. Then prove that T is continuous. Deduce that every linear transformation from N into any arbitrary normed linear space N' is continuous.

   b) If L is a normed linear space with respect to two norms $\|.\|$ and $\|.\|'$, then prove that these two norms are equivalent if and only if there exist two positive real numbers K_1 and K_2 such that $K_1\|x\| \le \|x\|' \le K_2\|x\|$, for all $x \in L$. **(10+4)**

4. a) If N is a normed linear space, then prove that there is an isometric isomorphism of N into N^{**}.

   b) Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M. If d is the distance from x_0 to M, then show that there exists a functional f_0 in N^* such that $f_0(M) = 0$, $f_0(x_0) = 1$ and $\|f_0\| = \frac{1}{d}$. **(7+7)**

5. a) Let B and B' be Banach spaces and T be a linear transformation of B into B'. Prove that T is continuous if and only if its graph is closed.

Contd...2

b) Let B be a Banach space and N be a normed linear space. If $\{T_i\}_{i\in I}$ is a nonempty set of continuous linear transformations of B into N such that $\{T_i(x)\}_{i\in I}$ is a bounded subset of N for each $x \in B$, then prove that $\{T_i\}_{i\in I}$ is a bounded subset of $\mathcal{B}(B,N)$. **(7+7)**

6. a) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a nonzero vector z_0 in H such that $z_0 \perp M$.

b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that $M+N$ is also closed.

c) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$. **(5+5+4)**

7. a) Let H be a Hilbert space and f be an arbitrary functional in H^*. Then prove that there exists a unique vector y in H such that $f(x) = \langle x, y\rangle$ for every $x \in H$.

b) Let H be a Hilbert space. Prove that the adjoint operation $T \mapsto T^*$ on $\mathcal{B}(H)$ satisfies:

i) $(T_1 + T_2)^* = T_1^* + T_2^*$ ii) $(T_1 T_2)^* = T_2^* T_1^*$.

c) If T is an operator on a Hilbert space H such that $\langle Tx, x\rangle = 0$ for all $x \in H$, then prove that $T = 0$. **(8+2+4)**

8. Let $\{e_1, e_2, ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. Then for any $x \in H$, show that $x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i \perp e_j$ for $j = 1,2,...,n$. Extend this result suitably to an arbitrary set $\{e_i\}_{i\in I}$ of orthonormal vectors in H. **(14)**

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PS 565.4

Reg. No. : | | | | | | |

St Aloysius College (Autonomous)
Mangaluru
Semester IV – P.G. Examination – M.Sc. MATHEMATICS
APRIL - 2018
PARTIAL DIFFERENTIAL EQUATIONS

Time: 3 Hours | Max. Marks: 70

Answer any FIVE FULL questions from the following: (14x5=70)

1.a) Show that the general solution of Lagrange's equation $P(x,y,u)u_x + Q(x,y,u)\, u_y = R(x,y,u)$ is $F(\phi(x,y,u), \psi(x,y,u)) = 0$, where F is an arbitrary function and $\phi(x,y,u) = c_1$ and $\psi(x,y,u) = c_2$ are integral curves of $\frac{dx}{P(x,y,u)} = \frac{dy}{Q(x,y,u)} = \frac{du}{R(x,y,u)}$

b) Find the orthogonal trajectories on the cone $yz + zx + xy = 0$ of the conics in which it is cut by the system of planes $x - y = c$ where c is a parameter. **(7+7)**

2.a) Prove necessary and sufficient condition that there exists between two functions $u(x,y)$ and $v(x,y)$ a relation $F(u,v) = 0$, not involving x or y explicitly is that $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

b) Test for integrability of $2y(a-x)\, dx + (z - y^2 + (a-x)^2)\, dy - y\, dz = 0$ and find its primitive. **(7+7)**

3.a) Find the general integral of the equation $(x-y)p + (y-x-z)q = z$ and the particular solution through the circle $z = 1,\ x^2 + y^2 = 1$.

b) Find the equation of the system of surfaces which cut orthogonally the cones of the system $x^2 + y^2 + z^2 = cxy$. **(7+7)**

4.a) Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and find their solution.

b) Find the characteristics of the equation $pq = z$. Also determine the integral surface which passes through the parabola $x = 0,\ y^2 = z$. **(7+7)**

5.a) Obtain the partial differential equation for the function $z = f\left(\frac{xy}{z}\right)$, where f is an arbitrary function.

b) Find the complete integral of $p^2z^2 + q^2 = 1$.

c) Find the complete integral of $(p^2 + q^2)y = qz$ using Charpit's method. **(4+3+7)**

Contd...2

6.a) Solve $(D + D')(D + 2D')u = x + y$.

b) Solve $(D - D' - 1)(D - D' - 2)u = e^{2x-y} + x$.

c) Solve $(D^2 + DD' - 6D'^2)u = ycosx$. (3+4+7)

7.a) Classify the following equation and reduce it to canonical form $u_{xx} + xu_{yy} = 0,\ x \neq 0, y \neq 0$.

b) Construct adjoint operator L^* for $Lu = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u$ where a, b, c are functions of x. (10+4)

8.a) A uniform rod of length L whose surface is thermally insulated initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.

b) A stretched string of finite length L is held fixed at its ends and is subjected to an initial displacement $u(x, 0) = u_0 \sin\left(\frac{\pi x}{L}\right)$. The string is released from this position with zero initial velocity. Find the resultant time dependent motion of the string. (7+7)

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PS 566.4

Reg. No: | | | | | | |

St Aloysius College (Autonomous)
Mangaluru
Semester IV – P.G. Examination - M. Sc. Mathematics
April - 2018

ALGEBRAIC NUMBER THEORY

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) Define Euler totient function $\phi(n)$. Prove that, for positive integers m,n;

   i) $\phi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right)$ and

   ii) $\phi(mn)=\phi(m)\phi(n)\frac{d}{\phi(d)}$ where $d=\gcd(m,n)$.

   b) Given a prime p, let $f(x)=c_0+c_1x+...+c_nx^n$ be a polynomial of degree n with integer coefficients such that $c_n \not\equiv 0 (\text{mod } p)$. Then prove that, the polynomial congruence $f(x)\equiv 0(mod\ p)$ has atmost n solutions.

   c) For any $a>n$, show that $n \mid \phi(a^n-1)$. **(8+4+2)**

2. a) Let p be an odd prime number greater than 3. Show that the numerator of $1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{p-1}$ is divisible by p^2.

   b) Define the Legendre symbol $(n|p)$. For an odd prime p prove that $(2|p)=(-1)^{\frac{p^2-1}{8}}$.

   c) Find all primes p such that $x^2\equiv 13(\text{mod } p)$ has a solution. **(4+5+5)**

3. a) State and prove Gauss Lemma.

   b) State and prove quadratic reciprocity law. Hence determine the value of Legendre symbol $(219|383)$. **(7+7)**

4. a) Prove that, the following statements are equivalent:

   i) α is an algebraic integer.

   ii) The minimal polynomial of α is monic and in $\mathbb{Z}[x]$.

   iii) $\mathbb{Z}[\alpha]$ is finitely generated $\mathbb{Z}$ - module.

   iv) $\exists$ a finitely generated $\mathbb{Z}$ - module $M \neq \{0\}$ of $\mathbb{C}$ such that $\alpha M \subseteq M$.

   b) Prove that every algebraic number field has an integral basis. **(6+8)**

5. Let O_K be the ring of algebraic integers of an algebraic number field K. Then prove that

   i) If I is a non-zero ideal of O_K, then I has an integral basis.

   ii) If $\alpha \in O_K$ then $Norm((\alpha))=|Norm_{K/Q}(\alpha)|$

   iii) If α is unit in O_K, then $Norm_{K/Q}(\alpha)=\pm 1$ **(14)**

Contd...2

6. a) Find the units in the ring of integers of an imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$.

b) Prove that the ring of integers of an imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$ is norm-Euclidean if $d = -1, -2, -3, -7$ or -11. **(6+8)**

7. a) Define a Dedekind domain. Prove that O_K is a Dedekind domain.

b) Show that every unique factorization domain is integrally closed.

c) Show that $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain, but not a principal ideal domain. **(8+4+2)**

8. a) Show that every non-zero ideal of O_K contains a product of finitely many non-zero prime ideals of O_K.

b) Define a fractional ideal of O_K. Show that any fractional ideal is finitely generated as an O_K - module.

c) Show that sum and product of two fractional ideals of O_K are again fractional ideals. **(6+4+4)**

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