Reg. No. | | | | | | |

PH 561.2

St Aloysius College (Autonomous)
Mangaluru
Semester II - P.G. Examinations - M.Sc. Mathematics
July 2022
ALGEBRA-II

Time : 3 Hours **Max. Marks : 70**

Answer any **FIVE FULL** questions from the following: (14 × 5 = 70)

1. (a) Prove that every Euclidean Domain (ED) is a Principal Ideal Domain(PID). Give an example of a PID which is not a ED.
   (b) Prove that every PID is a Unique Factorization Domain (UFD). (5+9)

2. (a) If R is a UFD, then prove that $R[x]$ is a UFD.
   (b) Is $\mathbb{Z}[x]$ a PID? Justify. (12+2)

3. (a) Prove that product of finite number of primitive polynomials in $\mathbb{Z}[x]$ is a primitive polynomial in $\mathbb{Z}[x]$.
   (b) Let $f(x) = a_nx^n + \cdots + a_1x + a_0$ be an integer polynomial and p be a prime integer such that $p \nmid a_n$. If the residue $\bar{f}$ of f modulo p is an irreducible element in $F_p[x]$, then prove that f is irreducible in $\mathbb{Q}[x]$. (6+8)

4. (a) Let K be an extension of a field F and $\alpha \in K$ be algebraic over F. If $p(x)$ is the minimal polynomial of α, then prove that $[F(\alpha) : F]$ is the degree of $p(x)$.
   (b) Let K be an extension of a field F and $\alpha, \beta \in K$ be algebraic over F. Then prove that there exists an F-isomorphism from $F(\alpha)$ to $F(\beta)$ which sends α to β if and only if α and β have the same minimal polynomial in $F[x]$. (5+9)

5. (a) Show that the set of all constructible real numbers form a subfield of $\mathbb{R}$ containing $\mathbb{Q}$.
   (b) If a real number α is constructible, then prove that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is a power of 2. (6+8)

6. (a) Prove that a regular pentagon is constructible.
   (b) Prove that any finite field has p^n elements, where p is a prime and n is a positive integer.
   (c) Given a prime p and a positive integer n, prove that there is a field of order p^n. (5+5+4)

7. (a) If p is a prime number such that a regular p-gon can be constructed with ruler and compass, then show that $p = 2^r + 1$ for some integer $r \geq 0$.
   (b) Prove that every finite extension of a field of characteristic zero has a primitive element. (5+9)

8. (a) If K is a finite extension of a field F of characteristic zero, then show that $O(G(K/F)) \leq [K : F]$.
   (b) State and prove the fundamental theorem of Galois theory. (6+8)

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PS 563.2

St Aloysius College (Autonomous)
Mangaluru
Semester II – P.G. Examination - M. Sc. Mathematics
JULY - 2022

RESEARCH METHODOLOGY AND ETHICS

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) What do you mean by research? Explain its significance in modern times.
   b) Distinguish between research methods and research methodology. **(7+7)**

2. a) Explain the qualities of good research.
   b) Write a short note on motivation in research.
   c) Write the techniques involved in defining a research problem. **(5+3+6)**

3. a) Describe the concept of literature review and discuss the various sources of literature.
   b) Write a short note on research objectives. **(10+4)**

4. a) Explain the types of research report.
   b) Explain the features of a research report.
   c) Write a short note on styles of bibliography. **(6+5+3)**

5. a) Discuss the properties of a mathematical definition.
   b) Explain the essential rules involved in mathematical writing. **(6+8)**

6. a) Explain the principles of research ethics.
   b) What are the advantages of research ethics? **(9+5)**

7. a) What does scientific misconduct refer to? Discuss the various forms of scientific misconduct.
   b) What is plagiarism? Describe the types of plagiarism. **(6+8)**

8. a) Explain the meaning and importance of IPR.
   b) Outline the concept of scholarly articles. Describe the steps involved in publication of scholarly articles. **(6+8)**

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St Aloysius College (Autonomous)
Mangaluru
Semester II - P.G. Examination - M.Sc. Mathematics
July - 2022
REAL ANALYSIS - II

Time: 3 hrs. **Max Marks: 70**

Answer any FIVE FULL questions from the following :

1. a) Let f be a bounded real function defined on $[a,b]$ and α be a monotonically increasing function on $[a,b]$. If P^* is a refinement of P, derive the relation between $U(P,f,\alpha)$ and $U(P^*,f,\alpha)$.

   b) Let f be a bounded real function on $[a,b]$, α be a monotonically increasing function on $[a,b]$. When do we say that $f \in \Re(\alpha)$ on $[a,b]$? If $f \in \Re(\alpha)$ on $[a,b]$ then prove that for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. Also prove that if $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ holds for some $\varepsilon > 0$ and for some partition P of $[a,b]$ then it holds for every refinement of P with the same ε.

   c) If f is continuous on $[a,b]$, then prove that $f \in \mathcal{R}$ on $[a,b]$.

   d) If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ then prove that $f + g \in \mathcal{R}(\alpha)$. **(3+5+3+3)**

2. a) Let f be a bounded real function on $[a,b]$, and $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \le f(x) \le M$ for all $x \in [a,b]$. Let φ be a continuous real function on $[m,M]$. If $h(x) = \varphi(f(x))$ on $[a,b]$, then prove that $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

   b) Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$ and α is monotonic, continuous at every point at which f is discontinuous. Prove that $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

   c) Define a rectifiable curve in $\mathbb{R}^n$. **(6+6+2)**

3. a) Define the notions of pointwise convergence and uniform convergence of a sequence $\{f_n\}$ of functions.

   b) Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then prove that $\{f_n\}$ converges uniformly to a function f and $f'(x) = \lim_{n\to\infty} f_n'(x),\ a \le x \le b$.

   c) Let $C(x)$ denote the class of all complex valued, continuous, bounded functions on a compact metric space X. Prove that $C(x)$ is complete metric space with respect to the metric, $\|f - g\| = sup\{|f(x) - g(x)| : x \in X\}$. **(2+7+5)**

4. a) Suppose K is compact, $\{f_n\}$ is a sequence of continuous functions that converge pointwise to a continuous function f on K and $f_n(x) \ge f_{n+1}(x)$ for all $\in K$, $n = 1,2,3,\ldots$. Then prove that $f_n \to f$ uniformly on K.

   b) Prove that there exists a real continuous function on the real line which is nowhere differentiable. **(6+8)**

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St Aloysius College (Autonomous)
Mangaluru
Semester II - P.G. Examinations - M.Sc. Mathematics
July 2022
LINEAR ALGEBRA-II

Time : 3 Hours Max. Marks : 70

Answer any FIVE FULL questions from the following: $(14 \times 5 = 70)$

1. (a) If V is a finite dimensional real vector space with positive definite bilinear form, then prove that V has an orthonormal basis.
   (b) Prove that the following properties of a real $n \times n$ matrix A are equivalent:
      (i) A represents dot product with respect to some basis of $\mathbb{R}^n$.
      (ii) There is an invertible matrix $P \in GL_n(\mathbb{R})$ such that $A = P^t P$.
      (iii) A is symmetric and positive definite. (8+6)
2. State and prove the Sylvester's law for symmetric forms on a real vector space V. (14)
3. (a) If V is a finite dimensional complex vector space with Hermitian form $\langle , \rangle$ then prove that there exists an orthonormal basis for V if and only if $\langle , \rangle$ is positive definite.
   (b) Let T be a linear operator on a Hermitian space V and let T^* be the adjoint operator. Then prove that T is Hermitian if and only if $\langle T(v), w\rangle = \langle v, T(w)\rangle$ for all $v, w \in V$. (9+5)
4. (a) If A is a hermitian matrix, then show that there exists a unitary matrix P such that PAP^* is a real diagonal matrix.
   (b) Let A be an $n \times n$ real symmetric matrix. Prove that e^A is symmetric and positive definite.
   (c) Show that eigen values of Hermitian operators are real. (6+6+2)
5. (a) Let M be a finitely generated R-module. Prove that M is isomorphic to a quotient of R^n for some $n \in \mathbb{N}$.
   (b) Let M be an R-module. Then prove that M is a free R-module if and only if M is isomorphic to R^n for some $n \in \mathbb{N}$.
   (c) Prove that any two bases of the same free module over a non-zero ring R have the same cardinality. (4+6+4)
6. (a) If A is an $m \times n$ matrix, prove that there exists products P, Q of elementary integer matrices such that QAP^{-1} is diagonal.
   (b) If $\phi : V \to W$ is a homomorphism of free abelian groups, then prove that there exist bases of V and W such that the matrix of ϕ has the diagonal form. (9+5)
7. (a) If G is a finitely generated free abelian group and H is a subgroup of G, show that there exists a basis $(u_1, \ldots, u_n)$ of G and a basis $(w_1, \ldots, w_m)$ of H such that

contd...2

(i) $m \leq n$

(ii) $w_i = d_i u_i$ for some positive integers d_i, $1 \leq i \leq m$

(iii) $d_1 | d_2 | \ldots | d_m$.

(b) Determine all integer solutions of the system $AX = 0$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (9+5)

8. (a) Prove that the following conditions on an R-module V are equivalent:

(i) every submodule of V is finitely generated

(ii) V satisfies the ascending chain condition.

(b) Let $\phi : V \to W$ be an R-module homomorphism. Prove the following:

(i) If $\ker \phi$ and $\mathrm{Im}\phi$ are finitely generated, then V is finitely generated.

(ii) If V is finitely generated and ϕ is surjective, then W is finitely generated.

(8+6)

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