PH 561.3

Reg No :

ST ALOYSIUS COLLEGE (AUTONOMOUS) MANGALURU

Semester III - P.G. Examination - M.Sc. Mathematics

November/December - 2023

COMPLEX ANALYSIS I

Time : 3 Hours | Max. Marks : 70

Answer **FIVE FULL** questions | (14x5=70)

1. a) Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{z}' = -1$. 5

   b) Derive a formula to find square roots of a non-zero complex number. Find the four values of $\sqrt[4]{-1}$. 4

   c) Show that $|a+b|^2 = |a|^2 + |b|^2 + 2Rea\bar{b}$ where $a, b \in \mathbb{C}$.Also show that if $a, b \in \mathbb{C}$ then $|a+b| = |a| + |b|$ if and only if $a\bar{b} \geq 0$. 5

2. a) State and prove Lucas theorem. 6

   b) Prove that for every power series with radius of convergence R, the series converges for all z with $|z| < R$. 8

3. a) Define the exponential function e^z and show that e^{iz} has a least positive period 2π and all other periods are integer multiples of 2π. 8

   b) Prove that $U(x, y) = x^2 - y^2 \, for \; all \; x \in \mathbb{R}$ is harmonic. Find the corresponding analytic function. 6

4. a) State and prove Morera's theorem. 5

   b) Evaluate $\int_{|z|=1} \frac{e^z}{z-a} dz$ 5

   c) Show that the index of a with respect to a piece-wise differentiable closed curve γ which does not pass through a is constant in each of the regions determined by γ 4

5. a) Show that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie on a circle or on a straight line 8

   b) If $p(x, y)$ and $q(x, y)$ are real or complex valued continuous functions defined in a region Ω and if γ is any curve in Ω then show that $\int_\gamma pdx + qdy$ depends only on the end points of γ if and only if there exists a function $u(x, y)$ in Ω with $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$. 6

6. a) When do you say that the points z and z^* are symmetric with respect to the circle through z_1, z_2 and z_3? Show that $w = \frac{z-1}{z+1}$ maps the imaginary axis in the z - plane onto the circle $|w| = 1$. 5

   b) Define length of an arc. Find the length of the arc $z = z(t) = t + i2t, \; t \in [1, 2]$. 4

   c) State the Cauchy's integral formula and compute $\int_{|z|=2} \frac{dz}{z^2+1}$. 5

7. a) If $\varphi(\xi)$ is continuous on an arc γ, show that $F_n(z) = \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n} d\xi$ is analytic in each of the regions determined by γ and its derivative is $F_n'(z) = nF_{n+1}(z)$. 8

   b) Define cross ratio. Let T be a linear transformation which fixes $1, 0, \infty$ show that T must be the identity map. 6

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8. a) If $f(z)$ is analytic on a rectangle R except at a point a in the interior of R and if $\lim_{z \to a}(z-a)f(z) = 0$, then show that $\int_{\partial R} f(z)dz = 0$. 5

b) State and prove open mapping theorem. Also state and prove maximal principal for analytic function 5

c) Find the value of $\int_{|z|=2} \frac{e^z}{z-1} dz$. 4

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TOPOLOGY

Time : 3 Hours Max. Marks : 70

Answer FIVE FULL questions (14x5=70)

1. a) Define and compare the standard, lower limit and the k- topology on the real line $\mathbb{R}$ 8

   b) If $\mathcal{B}$ and $\mathcal{B}'$ are bases for the topologies τ and τ' respectively on a set X, then show that τ' is finer than τ if and only if for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. 6

2. a) Define a closed set in a topological space X. Let Y be a subspace of X. Prove that a subset A of Y is closed in Y if and only if $A = C \cap Y$ for some closed set C in X. 6

   b) Show that A is the disjoint union of $\text{int}(A)$ and $\text{Bd}(A)$. Also prove that $\text{Bd}(A) = \varnothing$ if and only if A is closed. 8

3. a) Prove or disprove : Every T_1 - space is a T_2 - space. 4

   b) Prove that a space X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in the product space $X \times X$. 5

   c) Show that product of two Hausdorff spaces is Hausdorff. 5

4. a) Let $f : A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a)), \forall a \in A$. Then prove that f is continuous if and only if the coordinate functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous. 5

   b) Show that a continuous open map need not be closed. 4

   c) State and prove the pasting lemma. 5

5. Define a homeomorphism. If $f : X \to Y$ is a bijective continuous map where X is compact and Y is Hausdorff, then prove that f is a homeomorphism. 14

6. a) Define a separation for a topological spaces. Prove that a finite product of connected spaces is connected 8

   b) Prove that every open covering of a compact metric space has a Lebesgue number. 6

7. a) Define a separable space. Prove that every second countable space is separable. Show that the converse holds if X is metrizable 8

   b) Define a regular space. Show that every compact Hausdorff space is regular. 6

8. State and prove the Urysohn lemma. 14

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ST ALOYSIUS COLLEGE (AUTONOMOUS) MANGALURU

Semester III - P.G. Examination - M.Sc. Mathematics

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NUMERICAL ANALYSIS WITH COMPUTATIONAL LAB

Time : 3 Hours Max. Marks : 70

Answer **FIVE FULL questions** (14x5=70)

1. a) Find the largest eigen value in modulus and the corresponding eigen vector of the matrix using power method. 8

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$

b) Perform 3 iterations of the Bairstow method to extract the quadratic factor $x^2 + pq + q$ from the polynomial $P(x) = x^3 + x^2 - x + 2$. 6

2. a) Solve the equation $f(x) = x^3 - 5x + 1, x_0 = 0, x_1 = 0.5, x_2 = 1$ using Muller's method. 8

b) State and prove Braver theorem. 6

3. a) Solve the equation $f(x) = cosx - e^x$ using Bisection method, upto two decimal points. 6

b) Describe briefly Gauss -Jordan method. Find the inverse of the co-efficients matrix of the system $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ using Gauss Jordan method with pivoting and hence solve the system. 8

4. a) Derive Gregory-Newton backward difference interpolation formula. 8

b) For the following data calculate the forward and backward difference polynomials. Compute at $x = 0.5$ and $x = 0.35$. 6

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.0	2.28

5. a) Derive Composite Simpson's 1/3rd rule and Gauss Quadrature one-point formula. Also apply the Gauss Quadrature method to solve the integral $I = \int_1^2 \frac{2x}{1+x^4} dx$ and compare it with the original solution. 8

b) Given the following values of $f(x) = ln(x)$, find the approximte value of $f'(2.0)$ using linear and quadratic interpolation and $f''(2.0)$ using quadratic interpolation. 6

i	0	1	2
x_i	2.0	2.2	2.6

6. a) Evaluate the integral $\int_1^{1.5} \int_1^2 \frac{1}{x+y} dxdy$ using Simpson's 1/3rd rule with $h = 0.5$ along $x-$axis and $k = 0.25$ along y--axis. 6

b) Derive Lobatto 3-point formula. 8

7. a) State and prove Taylor series method and also derive its formula for error. 6

b) Using Adam-Bashworth's 3rd order method solve the initial value problem $u' = u^3 t, u(0) = 1$ on $[0, 1]$ with $h = 0.2$. 8

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8. a) Using Huen's method solve the initial value problem $u' = u^3 + t^2, u(0) = 1$. Estimate $u(0.4)$ when $h = 0.2$. 6

b) Solve the initial value problem $u' = u + t, u(0) = 1, h = 0.2$ in the interval $[0, 1]$ using fourth order R-K method. 8

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COMMUTATIVE ALGEBRA

Time : 3 Hours

Answer FIVE FULL questions — Max. Marks : 70

(14x5=70)

1. a) Show that only idempotents in a local ring are 0 and 1. 4

   b) Prove that every proper ideal in a ring is contained in a maximal ideal. 5

   c) Define the Jacobson radical $J(A)$ of a ring A. If U denotes the class of all units in A, then show that the Jacobson radical $J(A) = \{x \in A \,:\, 1 - xy \in U \;\forall\, y \in A\}$. 5

2. a) Define the prime spectrum $Spec(A)$ of a ring A. Prove that $Spec(A)$ is a compact topological space. 8

   b) If I and J are any two ideals of a ring A, then prove that $r(I+J) = r(r(I)+r(J))$. 6

3. a) Let $f : A \to B$ be a ring homomorphism. Define extended and contracted ideals. For the ideals I and J in A, Prove that 8

   1. $(IJ)^e = I^e J^e$
   2. $(I : J)^e \subseteq (I^e : J^e)$
   3. $r(I)^e \subseteq r(I^e)$
   4. $(I+J)^e = I^e + J^e$
   5. $(I \cap J)^e \subseteq I^e \cap J^e$.

   b) Define radical $r(I)$ of an ideal in a ring A. Show that $r(P^n) = P$ for any prime ideal P and $n \in \mathbb{N}$. 6

4. a) If M is a nonzero finitely generated $A-$module then prove that M is isomorphic to a quotient of A^n for some $n \in \mathbb{N}$. 4

   b) Let A be a ring and M be a finitely generated $A-$module and let I an ideal of A such that $IM = M$. Prove that there exists $x \equiv 1 \ (mod\ I)$ such that $xM = 0$. 5

   c) State and prove Nakayama's lemma. 5

5. a) Let $g : A \to B$ be a ring homomorphism and S be a multiplicatively closed set in A such that $g(s)$ is a unit in B for every $s \in S$. Prove that there is a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$, where $f : A \to S^{-1}A$ given by $f(a) = a.1 \ \ a \in A$. 6

   b) Let A be a ring and S be a multiplicatively closed set in A. Show that the ring of fractions $S^{-1}A$ is the zero ring if and only if $0 \in S$. 3

   c) If I, J are ideals of a ring A, with J finitely generated then show that $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$. 5

6. a) Let S be a multiplicatively closed subset of a ring A. Prove that there is a one-to-one correspondence between the set of all prime ideals of $S^{-1}A$ and the set of all prime ideals of A which do not meet S. 9

Contd.. 2

b) Let A be a ring and let M, N be a $A-$modules. For a prime ideal P let M_P denote the $S^{-1}A-$module $S^{-1}M$, where $S = A - P$. Prove that the following statements are equivalent: 5

1. $M = 0$
2. $M_m = 0$ for each maximal ideal m of A
3. $M_p = 0$ for each maximal ideal p of A

7. a) Define an exact sequence of modules. Let M, M' and M'' be $A-$modules, and S be a multiplicatively closed subset of A. If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M then show that $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact at $S^{-1}M$. 5

b) Let I be an ideal of a ring A, and let $S = 1 + I$. Show that S is a multiplicatively closed subset of A. Further, show that $S = 1 + I$ is contained in the Jacobson radical of $S^{-1}A$. 4

c) Describe the localization of a ring A at a prime ideal P of A. 5

8. a) Define a $P-$ primary ideal in a ring. Show that the intersection of finitely many $P-$ primary ideals is $P-$primary. 5

b) Define an irreducible ideal in a ring. In a Noetherian ring A prove that every ideal is a finite intersection of irreducible ideals. 5

c) Let Q be a $P-$primary ideal in a ring A, x an element of A. Then prove the following 4

1. if $x \in Q$ then $(Q : x) = (1)$
2. if $x \notin Q$ then $(Q : x)$ is $P-$primary, and therefore $r(Q : x) = P$

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