PH 561.2 Reg. No: | | | | | | |

# St Aloysius College (Autonomous)

## Mangaluru

## Semester II - P.G. Examination - M. Sc. Mathematics

## May/June -2023

## ALGEBRA II



Time: 3 Hours Max. Marks: 70

**Answer any FIVE FULL questions from the following**

1. a) Define a unique factorization domain. Prove that every principal ideal domain is a unique factorization domain.

   b) Prove that in an integral domain every prime element is irreducible. Does the converse hold? Justify **(9+5)**

2. a) State and prove Gauss lemma.

   b) Prove that the maximal ideals in a principal ideal domain are the principal ideals generated by irreducible elements.

   c) Prove that the polynomial ring $\mathbb{Z}[x]$ is a unique factorization domain. **(4+3+7)**

3. a) If $K$ is an extension field of $F$ and $\alpha \in K$, then show that $F[\alpha]$ is a field if and only if $\alpha$ is algebraic over $F$.

   b) Let $K$ and $L$ be an extensions of a field $F$. Let $\alpha \in L$ and $\beta \in K$ be algebraic over $F$. Prove that there exist an $F$-isomorphism from $F(\alpha)$ to $F(\beta)$ if and only if $\alpha$ and $\beta$ are the roots of the same irreducible polynomial over $F$. **(6+8)**

4. a) Factor $x^3 + x + 1$ in $F_p[x]$, when $p = 2, 3$ and 5.

   b) Is a regular $7-$ gon constructible? Justify

   c) Determine $[\mathbb{Q}(3\sqrt{2}, 4\sqrt{5}) : \mathbb{Q}]$. **(6+4+4)**

5. a) Prove that the set of all constructible numbers forms a subfield of $\mathbb{R}$ containing $\mathbb{Q}$.

   b) Let $p$ be a prime number and $n$ be a positive integer. Prove that there exists a field with $p^n$ elements. **(7+7)**

6. a) Find the splitting field and degree of extension of the splitting field of $f(x) = x^6 + x^3 + 1$ over $\mathbb{Q}$.

   b) State and prove the primitive element theorem. **(6+8)**

7. a) If $G$ is a finite group of automorphism of a field $K$ and $F$ is the fixed field of $G$ then, show that $O(G) = [K : F]$

   b) Let $G$ be a finite group of automorphisms of a field $K$. Let $F$ be a fixed field of $G$ and $(\beta_1, \beta_2, ..., \beta_r)$ be orbit of $\beta \in G$ under the action of $G$ on $K$. Prove that $(x - \beta_1)(x - \beta_2)...(x - \beta_r)$ is the minimal polynomial of $\beta$ and $r \mid O(G)$. **(7+7)**

8. a) State and prove the fundamental theorem of Galois theory.

   b) Let $K$ be a finite extension of a field $F$. Prove that $K$ is a Galois extension of $F$ if and only if $K$ is the splitting field of a separable polynomial with coefficients in $F$. **(9+5)**

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**Mangaluru**

**Semester II – P.G. Examination- M. Sc. Mathematics**



**May/June - 2023**

# REAL ANALYSIS - II

**Time: 3 Hours** **Max. Marks: 70**

**Answer any FIVE FULL questions from the following** **(14x5=70)**

1. a) Let $f$ be a bounded real function defined on $[a,b]$ and $\alpha$ be a monotonically increasing function on $[a,b]$. Define the upper and lower Riemann integrals of $f$ with respect to $\alpha$ over $[a,b]$. Prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

   b) If $f$ is continuous on $[a,b]$, then prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

   c) If $f$ is bounded real function on $[a,b]$ with $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ on $[a,b]$ then prove that $f \in \mathcal{R}(\alpha_1+\alpha_2)$ on $[a,b]$ and

   $$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 .$$ **(6+3+5)**

2. a) Suppose $\alpha$ is monotonically increasing on $[a,b]$ and $\alpha' \in \mathcal{R}$ on $[a,b]$. Let $f$ be bounded real valued function on $[a,b]$. Then show that $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$.

   b) If a curve $\gamma$ is continuously differentiable on $[a,b]$ then prove that $\gamma$ is rectifiable and that $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$. **(7+7)**

3. a) State and prove Cauchy criterion for uniform convergence of a sequence of functions.

   b) Let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}, \forall x \in \mathbb{R}, n = 1,2,....$ Show that the sequence $\{f_n\}$ converges to a function $f$ but $\{f_n'\}$ does not converge to $f'$.

   c) Let $\alpha$ be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ for $n = 1,2,3,...,$ and suppose $f_n \to f$ uniformly on $[a,b]$. Then prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$. **(4+4+6)**

4. a) Let $C(X)$ denote the set of all complex valued, continuous bounded functions on a metric space $X$. Show that $C(X)$ is a complete metric space with respect to the metric $\|f-g\| = \sup_{x\in X} |f(x)-g(x)|,\ f,g \in C(X)$.

   b) Prove that there exists a real continuous function on the real line which is nowhere differentiable. **(5+9)**

Contd...2

5. State and prove Stone's generalization of Weierstrass theorem. **(14)**

6. a) If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set $E$, then prove that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

b) Define equicontinuous family of functions on a set $E$. If $K$ is a compact metric space, $f_n \in C(K)$ for $n = 1,2,3,....,$ and if $\{f_n\}$ converges uniformly on $K$, then prove that $\{f_n\}$ is equicontinuous on $K$. **(7+7)**

7. a) Test the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

b) Let $\phi$ be bounded on $[a,\infty)$, integrable on $[a,t]$ for $t \geq a$. If $\int_a^\infty f(x)dx$ converges absolutely, then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent.

c) Prove that every absolutely convergent integral is convergent. **(3+7+4)**

8. a) State and prove the contraction principle.

b) State and prove the implicit function theorem. **(6+8)**

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# St Aloysius College (Autonomous)

## Mangaluru

## Semester II – P.G. Examination – M.Sc. MATHEMATICS

## May/June 2023

## RESEARCH METHODOLOGY AND ETHICS



**Time: 3 Hours** **Max. Marks: 70**

**Answer any FIVE FULL questions from the following:**

1.a) What do you mean by research? Explain its significance in modern times.

b) What are the characteristics of a good research problem?

c) How is fundamental research different from applied research? Elaborate.

**(6+3+5)**

2.a) What are the problems encountered by researchers in India?

b) Explain the importance of setting clear research objectives and discussing the role of motivation in research. Provide examples to support your answer.

**(8+6)**

3.a) Distinguish between research methods and research methodology.

b) How does a statement of the problem differ from research questions and objectives? **(7+7)**

4.a) What are the benefits of conducting a literature review in the research process, and how does it contribute to the overall research study?

b) Explain the different types of referencing styles in a research report.

**(7+7)**

5.a) What are the essential rules that should be followed while writing a mathematical document? How can one effectively write definitions, theorems, and proofs in mathematical writing?

b) Explain the role of TeX editor in a mathematical writing. **(8+6)**

6.a) Discuss the concept of research ethics and their significance in academic research. Provide examples of ethical misconduct in research and explain the consequences of such behaviour.

b) What are the advantages and limitations of adhering to research ethics?

**(7+7)**

7.a) What is meant by scientific misconduct, and what are the different types of scientific misconduct that researchers should be aware of?

b) How can universities and academic institutions promote awareness about the consequences of plagiarism among students and researchers? **(6+8)**

8.a) Describe the concept of Intellectual Property Rights. Why is it important for researchers and scholars to understand the concept of Intellectual Property Rights?

b) Write the meaning and importance of patents, copyrights and trademarks.

**(7+7)**

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# St Aloysius College (Autonomous)
## Mangaluru
## Semester II - P.G. Examinations - M.Sc. Mathematics

**May/June – 2023**



## LINEAR ALGEBRA-II

Time : 3 Hours Max. Marks : 70

**Answer any FIVE FULL questions from the following:** $(14 \times 5 = 70)$

1. (a) If $A$ is the matrix of a bilinear form with respect to a basis, then prove that the matrices $A'$ which represent the same form with respect to different bases are the matrices $A' = QAQ^t$ for some invertible matrix $Q$.
   (b) If $V$ is a finite dimensional real vector space with positive definite bilinear form, then prove that $V$ has an orthonormal basis.
   (c) Find an orthonormal basis for $\mathbb{R}^2$ with respect to the form $X^tAY$, where $A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$

   (4+6+4)

2. State and prove the Sylvester's law for symmetric form on a real vector space $V$. (14)

3. (a) Prove the Spectral theorem for a Hermitian operator on a Hermitian space.
   (b) Let $A$ be a real symmetric matrix. Prove that $e^A$ is symmetric and positive definite.

   (8+6)

4. (a) Let $T$ be a linear operator on a Hermitian space $V$ and let $T^*$ be the adjoint operator. Then show that
      (i) $T$ is Hermitian if and only if $\langle T(v), w\rangle = \langle v, T(w)\rangle$ for all $v, w \in V$.
      (ii) $T$ is unitary if and only if $\langle T(v), T(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$.
      (iii) $T$ is normal if and only if $\langle T(v), T(w)\rangle = \langle T^*(v), T^*(w)\rangle$ for all $v, w \in V$.
   (b) Find a unitary matrix $P$ so that $PAP^*$ is diagonal, where $A = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$
   (c) Prove that the eigen values of a Hermitian operator are real. (6+5+3)

5. (a) Let $R$ be a ring and $V$ be a free $R$-module of finite rank. Prove or disprove the following:
      (i) Every submodule of $V$ is free.
      (ii) Every set of generators contains a basis of $V$
   (b) State and prove Schur's lemma.
   (c) Let $R$ be a non-zero ring. Prove that any two bases of a free $R$-module have the same cardinality. (5+4+5)

Contd...

6. (a) Let $A$ be a non-zero $m \times n$ integer matrix. Then prove that there exists products $P, Q$ of elementary interger matrices such that

$$QAP^{-1} = \left[\begin{array}{cccc|c} d_1 & & & & \\ & d_2 & 0 & & \\ & & d_3 & & 0 \\ & 0 & & \ddots & \\ & & & d_r & \\ \hline & & 0 & & 0 \end{array}\right]_{m\times n}.$$

where each $d_i$ is a positive integer, $1 \leq i \leq r$, and each one divides the next: $d_1 \mid d_2 \mid \cdots \mid d_r$.

(b) Find a basis for the $\mathbb{Z}$-module of all integer solutions of the system $AX = 0$, when $A = \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix}$ (8+6)

7. (a) Prove that the following conditions on an $R$-module $V$ are equivalent:

(i) Every submodule of $V$ is finitely generated

(ii) $V$ satisfies the ascending chain condition.

(b) Let $\phi : V \to W$ be an $R$-module homomorphism. Prove that if $\ker \phi$ and $\text{Im}\phi$ are finitely generated, then $V$ is finitely generated.

(c) If $W$ is a submodule of an $R$-module $V$ such that $W$ and $V/W$ are finitely generated, then show that $V$ is finitely generated. (7+4+3)

8. (a) State and prove Structure Theorem for Abelian Groups. (14)

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**St Aloysius College (Autonomous)**

**Mangaluru**

**Semester IV – P.G. Examination - M. Sc. Mathematics**

**May / June 2023**

# ALGEBRAIC NUMBER THEORY

**Time: 3 Hours** **Max. Marks: 70**

**Answer any FIVE FULL questions from the following** **(14x5=70)**

1. a) State and prove Gauss Lemma.

   b) If $p$ and $q$ are distinct odd primes, then prove that

   $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}$



**(7+7)**

2. a) Prove or disprove: set of all algebraic numbers is uncountable.

   b) If $\alpha$ is a real algebraic number of degree $n > 1$, then show that there exists a positive constant $c(\alpha)$ such that for any rational number $p/q$ with $(p,q) = 1, q > 0$, the inequality $\left|\alpha - p/q\right| > \frac{c(\alpha)}{q^n}$ holds.

   c) Show that $\sum_{n=0}^{\infty} \frac{1}{10^{n!}}$ is transcendental.

**(4+5+5)**

3. a) Define Euler's totient function $\varphi$. Show that it is multiplicative.

   b) Prove or disprove: If $(m,n) = 1$, then $(\varphi(m), \varphi(n)) = 1$.

   c) If $(a,m) = 1$, then show that $a^{\varphi(m)} \equiv 1 (mod\ m)$. Determine the last two digits of $3^{2020}$.

**(6+2+6)**

4. a) Let $'p'$ be a prime and $f(x) = c_0 + c_1x + \cdots + c_nx^n$ be a polynomial with integer coefficients, such that $c_n \not\equiv 0 (mod\ p)$. Then prove that the polynomial congruence $f(x) \equiv 0 (mod\ p)$ has at most $n$ solutions.

   b) State and prove Wilson's theorem.

   c) If $p$ is an odd prime, then show that

   $1^2.3^2.5^2 \ldots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} (mod\ p).$

**(5+5+4)**

Contd...2

5. a) Let $K$ be an algebraic number field and $[K : \mathbb{Q}] = n$. If $\alpha \in K$ and $\sigma_1, \sigma_2, \dots \sigma_n$ are the distinct $\mathbb{Q}$-isomorphisms of $K$ into $\mathbb{C}$, then prove that

   i) $Tr_k(\alpha) = \sigma_1(\alpha) + \sigma_2(\alpha) + \cdots + \sigma_n(\alpha)$

   ii) $N_k(\alpha) = \sigma_1(\alpha)\,\sigma_2(\alpha) \dots \sigma_n(\alpha)$

   Further if $\alpha \in O_k$, then show that $Tr_k(\alpha)$ and $N_k(\alpha)$ are integers.

   b) Prove that every algebraic number field has an integral basis.

(7+7)

6. a) If $K = \mathbb{Q}(\sqrt{d})$, where $d$ is a square-free integer show that

$$O_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{d}, & if \ \ d \equiv 2 \ or \ 3 (mod\ 4) \\ \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{d}}{2}\right), & if \ \ d \equiv 1 (mod\ 4) \end{cases}$$

   b) Solve $y^2 + 2 = x^3$, for $x, y \in \mathbb{Z}$.

(8+6)

7. a) Let $K$ be an algebraic number field. If $I, J$ are non-zero ideals of $O_K$ with $I \subsetneq J$, then show that $N(I) > N(J)$.

   b) Define a Dedekind domain. Prove that $O_K$ is a Dedekind domain.

(4+10)

8. a) If $\wp$ is a prime ideal of $O_K$, then prove that $\wp^{-1}$ is a fractional ideal and $\wp\wp^{-1} = O_K$.

   b) Prove that every ideal in $O_K$ can be written as product of prime ideals uniquely.

(7+7)

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